***J. Indian Water Resour. Soc.,***
***Vol. 42, No. 2, Apr., 2022***

# GROUNDWATER FLOW SIMULATION USING MLPG MESHLESS METHOD

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## ABSTRACT

*Depletion of groundwater quality and quantity is a major challenge. Groundwater flow simulation requires a better understanding of the complex groundwater flow behavior. The flow behavior can be studied by solving the governing equation by analytical or numerical methods. As the analytical solutions involve many assumptions and are not applicable in cases of complex geometry, numerical methods are widely used. The conventional numerical methods such as Finite Difference Method and Finite Element Method are based on construction of mesh. However, the meshing and remeshing is cumbersome and is sometimes computationally inefficient. The Meshless methods were developed to resolve this problem. Various meshless methods have been developed in the past, such as Radial Point Collocation Method, Element Free Galerkin Method, etc. In this study, the Meshless Local Petrov Galerkin Method (MLPG) is adopted for the simulation of groundwater flow in a confined aquifer. The Improved Interpolating Moving Least Squares (IIMLS) scheme is implemented as the approximation function. A simulation model is developed in 2D using MATLAB for the solution of confined aquifer problems. The model is verified with the help of problems for which the analytical solutions are available. Further the model is applied for a field problem. This study shows that MLPG is an efficient method and gives accurate solutions.*

***Keywords:*** *Meshless models,Meshless weak form, groundwater contamination modeling, Meshless Local Petrov Galerkin Method, Improved Interpolating Moving Least Squares*

## INTRODUCTION

Groundwater models are necessary for understanding the aquifer system and for predicting the changes in the system. With the advent of computers, numerical groundwater models became a powerful tool in groundwater modeling. The direct analytical solutions are suitable for simple problems involving homogeneous, isotropic aquifers and simple geometry. However, in reality, these assumptions are not valid and, in some cases, analytical methods fail. In order to take into account, the complexity of the system, a numerical method can be used.Numerical models convert the set of governing equations, which are in the form of partial derivative equations, into a set of algebraic equations, which are then solved. Traditional numerical methods which are proven to be efficient for groundwater modelling include Finite Difference Method (FDM), Finite Element Method (FEM). But, these methods are based on a mesh and meshing and remeshing process is cumbersome and computationally expensive. This led to the development of meshless methods, in which these limitations are overcome.

Various meshless methods have been presented in the recent years. Based on the formulation, meshless methods can be broadly classified as weak forms and strong forms [6]. Meshless weak forms have an advantage that the implementation of derivative boundary conditions is direct. The governing equation for groundwater flow is a partial differential equation [4], which is converted into a set of algebraic equations for solving. Various meshless methods were discussed by Liu and Gu [6]. A truly meshless method, the Meshless Local Petrov Galerkin Method (MLPG) was first proposed by Atluri and Zhu [2]. This method does not require any background mesh for the purpose of integration and interpolation. This method uses overlapping local support domains.

For groundwater flow simulation studies, Mategaonkar and Eldho [7] developed the strong form polynomial point collocation method with multiquadric radial basis function, Guneshwor et al. [5] developed meshless radial point collocation method. Swathi and Eldho [9] applied MLPG with exponential radial basis function for flow problems. Wang et al. [11] developed the Improved Interpolating Moving Least Squares technique (IIMLS), in which the limitations of Moving Least Squares (MLS) scheme, such as difficulty in applying essential boundary conditions, is overcome.

In this paper, the MLPG method with Improved Interpolating Moving Least Squares (IIMLS) approximation is used for groundwater flow simulation. The numerical model is developed in MATLAB and is verified for hypothetical problems. Further, the proposed model is applied to a field case study.

### Governing Equation and Meshless Formulation

For a heterogeneous, anisotropic confined aquifer, the governing equation is given by [4]:

$$\frac{\partial}{\partial x}\left(T_x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T_y \frac{\partial h}{\partial y}\right) = \left(S\frac{\partial h}{\partial t}\right) \pm \sum_{w=1}^{W} Q_w \delta(x - x_w)(y - y_w) - f \quad (1)$$

The initial condition can be taken as:

$$h(x, y, 0) = h_0(x, y) \qquad (x, y) \in \Omega \quad (2)$$

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*Manuscript No. 1586*

*Received : 16 July, 2022;   Accepted : 12 November, 2022*

The boundary conditions (BC) are of two types: the constant head boundary condition (essential or Dirichlet boundary condition) and constant flux boundary condition (natural or Neumann boundary condition):

$h(x, y, t) = h_1(x, y, t) \quad (x, y) \in \Gamma_1$ (Dirichlet BC)

$T\frac{dh}{dn} = q(x, y, t) \quad (x, y) \in \Gamma_2$ (Neumann BC) (3)

Here, $T_x$ and $T_y$ represent transmissivity in x and y directions respectively, S is the storage coefficient, δ is Dirac delta function, $Q_w$ is the source/sink (negative sign for source and positive sign for sink),($x_w$, $y_w$) and Warethe location and the total number of source/sink,fis known inflow rate, t is time,Ω is the flow domain, Γ is the domain boundary and suffixes 1 and 2 represent essential and natural boundaries, $h_0$ is the initial head in the aquifer, T may be $T_x$ or $T_y$,and n is the normal vector of the boundary, $h_1$ is constant head on Dirichlet boundary and q is inflow at Neumann boundary.

In MLPG, which is a weak form method, the governing partial differential equation is multiplied by a suitable test function and then integrated. In this study, MLPG5 method is used [3], therefore, the test function is Heaviside step function. For a homogeneous confined aquifer, the equation (1) can be written as,

$$\nabla(T\nabla h) = \sum_{w=1}^{W} Q_w \delta(r - r_w) - f + S\frac{\partial h}{\partial t} \quad (4)$$

Here, r is position vector, $\delta(r - r_w)$ is Dirac delta function, which is equal to zero if $r \neq r_w$ and $\int_{\Omega} \boldsymbol{\delta(r - r_w) d\Omega = 1}$ for $r_w \in \Omega$.

Let υ be the test function. Multiplying the governing equation (4) by υ and integrating it over the sub-domain $\Omega_s$:

$$\int_{\Omega_s} \nabla(T\,\nabla h)\,\upsilon d\Omega = \int_{\Omega_s} \sum_{w=1}^{W} Q_w \delta(r - r_w)\upsilon d\Omega - \Omega s f \upsilon d\Omega + \Omega s S \partial h \partial t \upsilon d\Omega \quad (5)$$

Using the divergence theorem to first term of equation (5),

$$\int_{\Omega_s} \nabla(T\,\nabla h)\,\upsilon\, d\Omega = \int_{\partial\Omega_s} (T\nabla h) n \upsilon d\Omega - \int_{\Omega_s} (T\nabla h)\nabla\upsilon d\Omega \quad (6)$$

Substituting equation (6) and the Heaviside step function values (υ = 1 for r in $\Omega_s$ and 0 outside $\Omega_s$ and ∇υ = 0) in equation (5),

$$\int_{\partial\Omega_s} (T\,\nabla h)\, n\, d\Omega = \sum_{w=1}^{W} Q_w - \int_{\Omega_s} f d\Omega + \int_{\Omega_s} S\frac{\partial h}{\partial t} d\Omega \quad (7)$$

where, $\partial\Omega_s$ represents the boundary of $\Omega_s$.

Assuming that, within a sub domain, T and f remain constant and maximum of single source is active at a time, equation (7) can be reduced to,

$$T_j \int_{\partial\Omega_{s,j}} \frac{\partial h}{\partial n} d\Omega = Q_j - f_j A_{\Omega_s} + \int_{\Omega_s} S\frac{\partial h}{\partial t} \vartheta d\Omega \quad (8)$$

where, j represents node, $A_{\Omega_s}$ is the area of the sub domain, $\frac{\partial h}{\partial n} = \nabla h n = \frac{\partial h}{\partial x} n_x + \frac{\partial h}{\partial y} n_y$.

In this study, circular sub-domains are used which are centered at nodes. Nodes can be distributed or irregularly. Some nodes are placed at the source points. Considering N as the total number of nodes, the head can be estimated as [3]:

$$h = \sum_{i=1}^{N} \emptyset_i \widehat{h_i} \quad (9)$$

where$\hat{h}_i$ is the fictitious head and $\Phi_i(r)$ is the shape function.

The summation in equation (9) is extended to support sub-domain which is different from the sub-domain. The sub-domain radius is taken as a small value, to maintain the local character of problem and the support sub domain must be larger, to estimate with respect to many surrounding nodes [6].

The dependent variable h and its derivatives are computed as:

$$\frac{\partial h}{\partial x} = \sum_{i=1}^{N} \frac{\partial \emptyset_i}{\partial x} \widehat{h_i} \ and \ \frac{\partial h}{\partial y} = \sum_{i=1}^{N} \frac{\partial \emptyset_i}{\partial y} \widehat{h_i} \quad (10)$$

Substituting equation (10) in (8),

$$\frac{\partial h}{\partial n} = \nabla hn = \frac{\partial h}{\partial x} n_x + \frac{\partial h}{\partial y} n_y = \frac{\partial \emptyset_i}{\partial x} \widehat{h_i}\, n_x + \frac{\partial \emptyset_i}{\partial y} \widehat{h_i}\, n_y \quad (11)$$

Studies of Pinder and Gray [8] and Wang and Anderson [10], proved that finite difference method is the best for time discretization [9]. Hence, Crank Nicolson scheme with weight factor θ = 0.5 is adopted in this study. For a function G, the time discretization can be thus done as:

$$\frac{\partial \hat{h}}{\partial t} = \frac{\hat{h}_{t+1} - \hat{h}_t}{\Delta t} = \theta[G^{t+1}(h, x, y)] + (1 - \theta)[G^t(h, x, y)] \quad (12)$$

Substituting equations (12) and (10) in (8),

$$\left(T_j \int_{\partial\Omega_{s,j}} \left(\frac{\partial \emptyset_i}{\partial x} \hat{h}_{t+1} n_x + \frac{\partial \emptyset_i}{\partial y} \hat{h}_{t+1} n_y\right) d\Omega\right)\theta + \left(T_j \int_{\partial\Omega_{s,j}} \left(\frac{\partial \emptyset_i}{\partial x} \hat{h}_t n_x + \frac{\partial \emptyset_i}{\partial y} \hat{h}_t n_y\right) d\Omega\right)(1 - \theta) = Q_j - f_j A_{\Omega_s} + \int_{\Omega_s} S\left(\frac{\hat{h}_{t+1} - \hat{h}_t}{\Delta t}\right) d\Omega \quad (13)$$

On rearranging the terms of equation (13),

$$\left(\left(T_j \int_{\partial\Omega_{s,j}} \left(\frac{\partial \emptyset_i}{\partial x} n_x + \frac{\partial \emptyset_i}{\partial y} n_y\right) d\Omega\right)\theta - \int_{\Omega_s} \frac{S}{\Delta t} d\Omega\right)\left(\hat{h}_{t+1}\right) = Q_j - f_j A_{\Omega_s} - \left(\left(T_j \int_{\partial\Omega_{s,j}} \left(\frac{\partial \emptyset_i}{\partial x} n_x + \frac{\partial \emptyset_i}{\partial y} n_y\right) d\Omega\right)(1 - \theta) - \Omega s S \Delta t\, d\Omega h t \quad (14)$$

Thus, the system of equations can be written as:

$$K\{\hat{h}_{t+1}\} = F \quad (15)$$

In equation (15),

$$K = \left( \left( T_j \int_{\partial\Omega_{s,j}} \left( \frac{\partial \phi_i}{\partial x} n_x + \frac{\partial \phi_i}{\partial y} n_y \right) d\Omega \right) \theta - \int_{\Omega_s} \frac{S}{\Delta t}\, d\Omega \right) \quad (16)$$

$$F = Q_j - f_j A_{\Omega_s} - \left( \left( T_j \int_{\partial\Omega_{s,j}} \left( \frac{\partial \phi_i}{\partial x} n_x + \frac{\partial \phi_i}{\partial y} n_y \right) d\Omega \right) (1 - \theta) - \Omega s S \Delta t \, d\Omega h t \quad (17)$$

The solution can be computed by solving the matrix. Then iteratively, the right side of F vector acts as the new F vector. By the solution, $\hat{h}_{t+1}$ which is used in the next iteration is found. For each main node inside the domain, equation (17) can be used to build the linear equation in the unknown $\hat{h}_t$. The integral is computed for the circumference by Gauss Quadrature method.

A constant head boundary is applied using the penalty method, with α as penalty function:

$$\propto \int_{\Gamma_1} (h - h_i) d\Gamma = 0 \quad (18)$$

A no flow boundary condition can be applied as:

$$T_j \int_{\Gamma_2} \frac{\partial h}{\partial n}\, d\Gamma = -q_j A_{\Omega_s} \quad as \ \frac{\partial h}{\partial n} = 0 \ in \ \Gamma_2 \quad (19)$$

The Neumann boundary condition with a constant normal flux can be applied as:

$$T_j \int_{\Gamma_2} \frac{\partial h}{\partial n}\, d\Gamma + q = -q_j A_{\Omega_s} \quad as \quad T_j \int_{\Gamma_2} \frac{\partial h}{\partial n}\, d\Gamma = q \quad (20)$$

In the case of confined heterogeneous and/or anisotropic aquifers, the values of transmissivities at different zones are considered andthe transmissivities in x and y directions are substituted.

**Model Development and Verification**

The groundwater flow model is developed based on the above formulation. The steps involved in the model development are given below:

**Step 1:** The input values of transmissivity, porosity, boundary conditions, aquifer dimensions, recharge details etc., are taken and the time step and total time are fixed.

**Step 2:** The number of divisions is decided. The pattern of nodes (regular or irregular) is decided. The number of additional nodes required per source point is fixed. The size of support domain and sub domain are also fixed.

**Step 3:** The sub domains are constructed for all the nodes. The formulation is done in the code.

**Step 4:** The head values at all the nodes are obtained. For transient analysis, the head from previous time step is used as initial condition for current time step. The procedure is repeated till the final time step is reached.

Fig. 1 shows the flow chart of the MLPG model for the groundwater flow model in the confined aquifer.

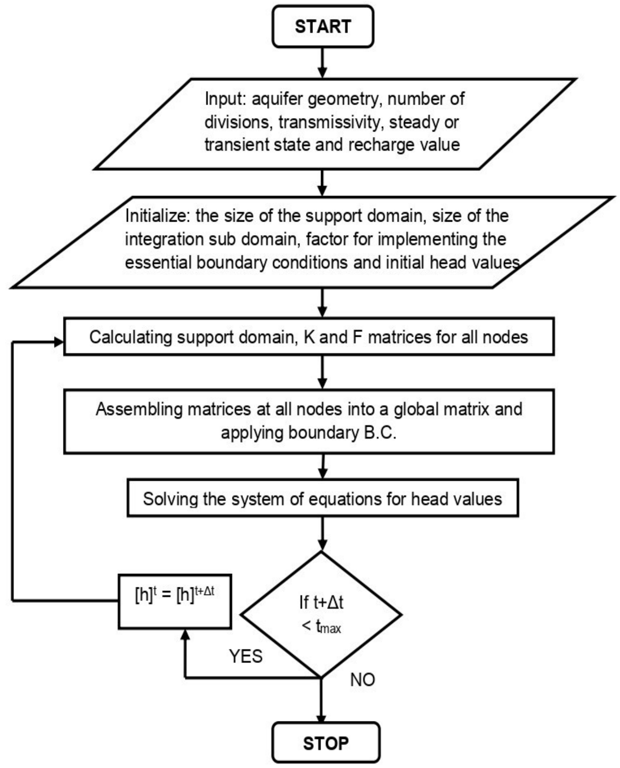



**Fig. 1. Flowchart for MLPG based Flow Model for a Confined Aquifer**

The developed flow model is applied to a hypothetical confined aquifer and the results are compared with the FEM results given by Willis and Yeh [12]. The problem domain is shown in Fig. 2. In the Fig., Lx and Ly are the domain size in the x and y directions respectively. h is the head. The aquifer domain has dimensions of 1400 m × 1400 m. A constant head of 100 m is considered on the right and left boundaries and the top and bottom boundaries are assumed no flux boundaries. The transmissivity value is 100 $m^2$/day and the Storativity is 0.001. Initial head for the simulation of the transient state problem is taken as 100 m. A pumping well is located at the center of the aquifer and the pumping rate is 10000 $m^3$/day.

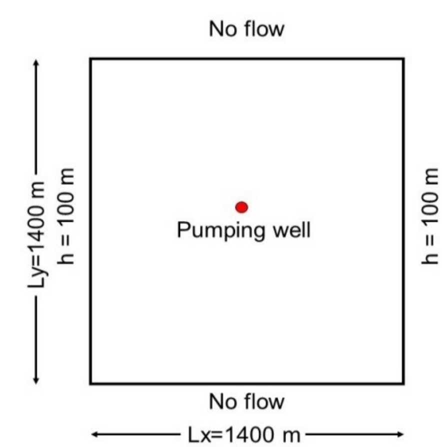



**Fig. 2. Aquifer domain for hypothetical case study**

**Table 1. Comparison of head values obtained from MLPG and FEM with the analytical solution for hypothetical case study**

| Node number | Analytical (m) | FEM (m) | Percentage difference | MLPG (m) | Percentage difference |
|---|---|---|---|---|---|
| 15 | 100 | 100 | 0 | 100 | 0 |
| 29 | 97.013 | 96.993 | 0.0021 | 96.875 | 0.142 |
| 43 | 93.804 | 93.768 | 0.038 | 93.645 | 0.169 |
| 57 | 90.095 | 90.051 | 0.049 | 89.961 | 0.148 |
| 71 | 85.451 | 85.413 | 0.044 | 85.327 | 0.145 |
| 85 | 78.983 | 78.974 | 0.011 | 78.853 | 0.088 |
| 99 | 67.953 | 67.762 | 0.281 | 67.765 | 0.277 |

The bottom left corner is taken as origin. The number of divisions in horizontal and vertical directions is 14 each. The nodes are numbered vertically, i.e., the point (0, 6000) is node number 15. Observation wells are located at the nodes 29, 43, 57, 71, 85 and 99. Fig. 3 shows the nodal arrangement along with the location of pumping and observation wells.

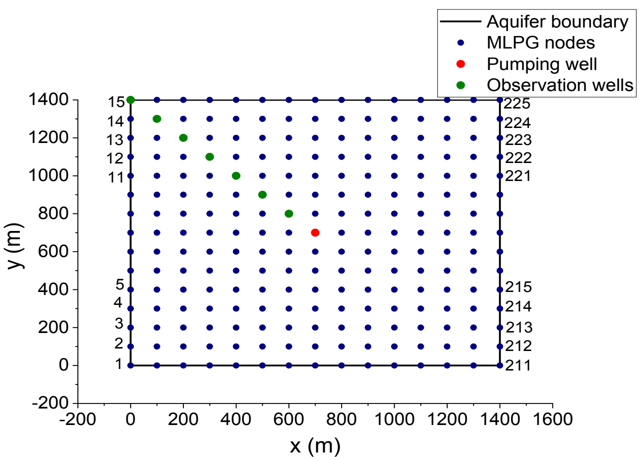



**Fig. 3: Nodal arrangement for hypothetical case study**

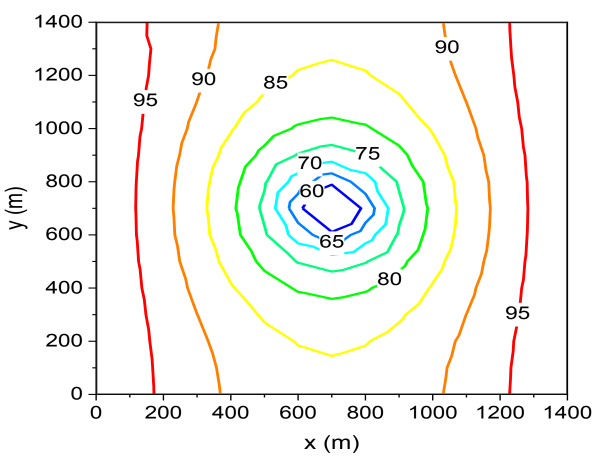



**Fig. 4: Contour plot for head distribution in hypothetical case study**

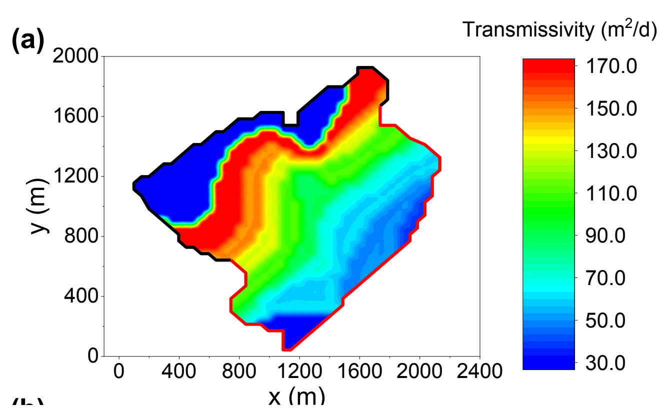
The results of the MLPG flow model are compared with the FEM results given by Willis and Yeh [12] and Mategaonkar and Eldho [7]. The simulation time is 10 days with a time step of 0.2 days. The radius of support domain is taken as twice the distance between the nodes. The total number of nodes is 225. The contour plot of the interpolated head values is illustrated in Fig. 4. The comparison of head values at observation wells obtained from MLPG and FEM is shown in Table 1. The results are sufficiently accurate. These results are greatly affected by the parameters like the size of the support domain and the size of the integration sub domain.

**Field Case Study**

The developed MLPG groundwater flow model is tested for a field problem. The data is taken from Anshuman et al. [1] and Guneshwor et al. [5]. The study area is located in Gujarat, India. The area of the aquifer is 4.5 km$^2$ and the aquifer is a heterogeneous confined aquifer [5]. There are 11 different transmissivity zones with the transmissivity value varying from 30 m$^2$/day to 170 m$^2$/day. The north, south, north-east and south west boundaries are bounded by a lake, with head values varying from 45 m to 47.5 m. There is a recharge zone in the aquifer which recharges at 0.045 m/day. The aim of this study is to calculate the head values in the aquifer and compare the results with the available FEM solution.

Fig. 5a shows the aquifer domain and the transmissivity zones. Fig. 5b depicts the nodal arrangement. A steady state model is developed for head variation. 1008 nodes are used all over the boundary. In this study, for MLPG method, the integration is done using Gauss Quadrature. θ value of equation (17) is taken as 0.5.

The factor for support domain is taken as 3.5. Fig. 6 shows the MLPG head values over the entire domain. Table 2 shows the comparison of the interpolated head values with the results from FEM and the results are found to be in agreement. The results of FEM are taken from Guneshwor et al. [5]. The MLPG solutions are very well matching with the FEM results, with a maximum difference of 0.69% between the results.



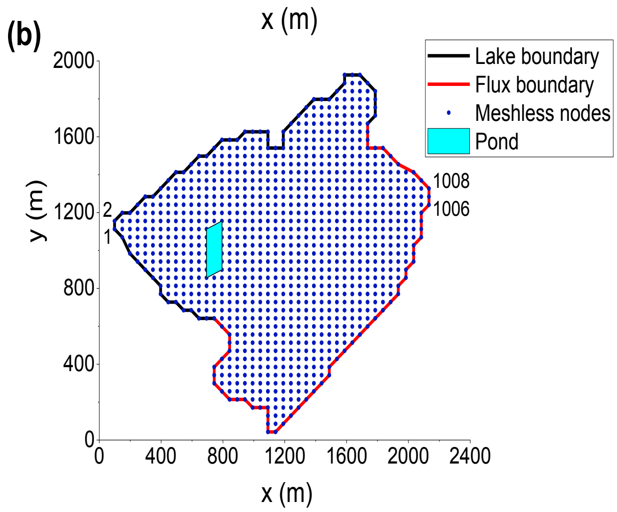



**Fig. 5. (a) Problem domain and zones of transmissivity (b)Nodal arrangement for field case study**

**Table 2. Comparison of results of field case study obtained from MLPG and FEM models**

| Node | Head from FEM (m) | Head from MLPG (m) | Percentage difference |
|---|---|---|---|
| 448 | 46.8 | 46.9 | 0.18 |
| 558 | 46.8 | 47.0 | 0.51 |
| 662 | 46.5 | 46.6 | 0.32 |
| 552 | 46.5 | 46.4 | 0.14 |
| 670 | 46.8 | 47.1 | 0.69 |
| 730 | 46.5 | 46.7 | 0.39 |

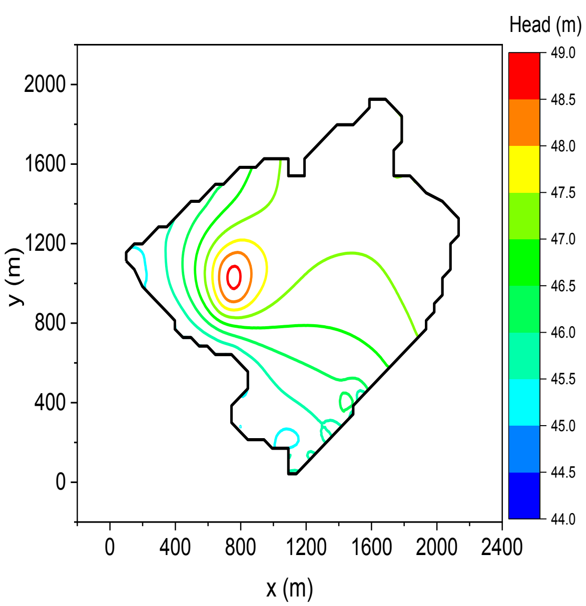



**Fig 6.** Contour plot of heads for field case study

## DISCUSSION

Meshless methods resolve the problem of meshing and remeshing in the traditional FDM, FEM methods. The whole domain is represented by a set of nodes, which do not require a background mesh. MLPG method has many advantages. It is a truly meshless method and no background mesh is required in stages of integration and interpolation. The implementation of the derivative boundary conditions is simple and direct. The solutions are stable and accurate. Local sub domains are used for integration in this method and the formulation is completely local.

The MLPG method was used for groundwater flow modeling. The developed model was verified with a hypothetical case study, in transient state condition and the results obtained were satisfactory. The model was applied to a real field aquifer [5] and the results were similar to the results obtained from FEM. Hence, MLPG can be successfully used in groundwater flow studies for larger field problems.

## CONCLUSIONS

In this study, a Meshless Local Petrov Galerkin (MLPG) method with IIMLS scheme was developed for the groundwater flow modelling. MLPG being a truly meshless method, the pre-processing efforts have been considerably reduced when compared to FDM and FEM. The developed model was tested on a hypothetical problem and a real field problem, and the results were accurate in both the cases. The MLPG solutions are in excellent agreement with the FEM solutions. However, the model results are highly dependent on the size of the support domain. This dependency is to be further studied. The meshless model proposed here can be efficiently used for large scale field problems.

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